Recursion - 4

28 October 2020 14:41

Problems on Homo Linear Recc. Relation with Constant coefficients using Characteristic Equation.

Q1. Solve the recurrence relation

$$f(n) = 7f(n-1) - 10f(n-2)$$

given that $f(0) = 4$ and $f(1) = 17$

sol:→ The given recc. relation

$$f(n) - 7f(n-1) + 10f(n-2) = 0 \quad \text{——— (1)}$$

with $f(0) = 4$ and $f(1) = 17$

is homo Linear recc. relation with const coeff of order 2.

For char eqn, take $f(n) = a^n$ in (1)

$$a^n - 7a^{n-1} + 10a^{n-2} = 0$$

$$\Rightarrow a^2 - 7a + 10 = 0$$

$$\Rightarrow (a-5)(a-2) = 0$$

$$\Rightarrow a = 2, 5$$

$$\therefore f(n) = A(2)^n + B(5)^n$$

As $f(0) = 4$ and $f(1) = 17$

$A(2)^0 + B(5)^0 = 4$ $\qquad$ $A(2)^1 + B(5)^1 = 17$

$A + B = 4$ $\qquad$ $2A + 5B = 17$

Now

$$A + B = 4$$
$$2A + 5B = 17$$

on solving $A = 1$, $B = 3$

Therefore, $f(n) = A(2)^n + B(5)^n$

$= 1(2)^n + 3(5)^n$

$= 2^n + 3(5)^n$

Q2:- Solve the recurrence relation

$$f(k) - 8f(k-1) + 16f(k-2) = 0$$

where $f(2) = 16$ and $f(3) = 80$.

Sol:→ Given recc. relation is homo. Linear recc. relation with const. coeffs of order 2.

$$f(k) - 8f(k-1) + 16f(k-2) = 0 \quad \text{——— (1)}$$

where $f(2) = 16$ and $f(3) = 80$

For char eqn, take $f(k) = a^k$ in (1)

$$a^k - 8a^{k-1} + 16a^{k-2} = 0$$

$\Rightarrow a^2 - 8a + 16 = 0$

$\Rightarrow (a-4)^2 = 0$

$\Rightarrow a = 4, 4$

$\therefore f(k) = (A_1 + kA_2)(4)^k$

As $f(2) = 16$ and $f(3) = 80$

$(A_1 + 2A_2)4^2 = 16$ $\qquad$ $(A_1 + 3A_2)4^3 = 80$

$A_1 + 2A_2 = 1$ $\qquad$ $(A_1 + 3A_2)4 = 5$

$4A_1 + 12A_2 = 5$

Now

$$A_1 + 2A_2 = 1$$
$$4A_1 + 12A_2 = 5$$

On solving $A_1 = \frac{1}{2}$, $A_2 = \frac{1}{4}$

Therefore $f(k) = (A_1 + kA_2)4^k$

$$= \left(\frac{1}{2} + k\frac{1}{4}\right)4^k$$

$$= \left(\frac{2+k}{4}\right)4^k$$

$$= (2+k)4^{k-1}$$

Q:→ Solve the recurrence relation

$$f(n) - f(n-3) = 0$$

Sol:→ Given recc. relation is homo linear recc. relation with constant coeff of order 3.

$$f(n) - f(n-3) = 0 \quad \text{———} \quad (1)$$

Char eqn (take $f(n) = a^n$)

$$a^n - a^{n-3} = 0$$

$$\Rightarrow a^3 - 1 = 0$$

$$\Rightarrow (a-1)(a^2+a+1) = 0$$

$$\Rightarrow a = 1 \quad , \quad a^2 + a + 1 = 0$$

$$a = \frac{-1 \pm \sqrt{1^2 - 4(1)(1)}}{2(1)}$$

$$= \frac{-1 \pm i\sqrt{3}}{2}$$

$$a = 1, \frac{-1+i\sqrt{3}}{2}, \frac{-1-i\sqrt{3}}{2}$$

$$f(n) = A(1)^n + B\left(\frac{-1+i\sqrt{3}}{2}\right)^n + C\left(\frac{-1-i\sqrt{3}}{2}\right)^n$$

Q:→ Solve the recurrence relation

$$T(k) - 7T(k-2) + 6T(k-3) = 0$$

where $T(0) = 8$, $T(1) = 6$ and $T(2) = 22$.

sol:→ Given recc. relation

$$T(k) - 7T(k-2) + 6T(k-3) = 0 \quad \text{——— (1)}$$

where $T(0) = 8$, $T(1) = 6$, $T(2) = 22$

For char eqn, take $T(k) = a^k$

$$a^k - 7a^{k-2} + 6a^{k-3} = 0$$

$$\Rightarrow a^3 - 7a + 6 = 0$$

$\pm 1, \pm 2, \pm 3, \pm 6$

$a = 1$ ✓

$1 - 7 + 6 = 0$

1	1	0	-7	6
	↓	1	1	-6
	1	1	-6	0

$$a^2 + a - 6 = 0$$

$$(a+3)(a-2) = 0$$

$$a = 2, -3$$

$$\therefore a = 1, 2, -3$$

$$T(k) = A(1)^k + B(2)^k + C(-3)^k$$

As $T(0) = 8$, $T(1) = 6$, $T(2) = 22$

$A + B + C = 8$, $A + 2B - 3C = 6$, $A + 4B + 9C = 22$

Now

$$A + B + C = 8$$
$$A + 2B - 3C = 6$$
$$A + 4B + 9C = 22$$

Augmented Matrix = $\left[\begin{array}{ccc:c} 1 & 1 & 1 & 8 \\ 1 & 2 & -3 & 6 \\ 1 & 4 & 9 & 22 \end{array}\right]$

$R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & 8 \\ 0 & 1 & -4 & -2 \\ 0 & 3 & 8 & 14 \end{array}\right]$$

$R_3 \to R_3 - 3R_2$

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & 8 \\ 0 & 1 & -4 & -2 \\ 0 & 0 & 20 & 20 \end{array}\right]$$

$$A + B + C = 8 \Rightarrow A = 5$$
$$B - 4C = -2 \Rightarrow B = 2$$
$$20C = 20 \Rightarrow C = 1$$

Therefore

$$T(k) = A(1)^k + B(2)^k + C(-3)^k$$
$$= 5(1)^k + 2(2)^k + 1(-3)^k$$
$$= 5 + 2^{k+1} + (-3)^k \quad \underline{\text{Ans}}$$

Q:→ Solve the recurrence relation

$9y(n+2) - 6y(n+1) + y(n) = 0$

sol:→ Given recc. relation is homo linear recc. relation of order 2 with constant coeffs

$$9y(n+2) - 6y(n+1) + y(n) = 0 \quad \text{—— (1)}$$

For char eqn, take $y(n) = a^n$ in (1)

$$9a^{n+2} - 6a^{n+1} + a^n = 0$$

$$\Rightarrow 9a^2 - 6a + 1 = 0$$

$$\Rightarrow (3a-1)^2 = 0$$

$$\Rightarrow 3a - 1 = 0, \quad 3a - 1 = 0$$

$$a = \frac{1}{3}, \frac{1}{3}$$

$$\therefore y(n) = (A_1 + nA_2)\left(\frac{1}{3}\right)^n$$

Q:→ Solve the recc. relation

$$y(n) - 3y(n-1) + 3y(n-2) - y(n-3) = 0$$

sol:→ Char eqn $(\because y(n) = a^n)$

$$a^n - 3a^{n-1} + 3a^{n-2} - a^{n-3} = 0$$

$$\Rightarrow a^3 - 3a^2 + 3a - 1 = 0$$

$$\Rightarrow (a-1)^3 = 0$$

$$\Rightarrow a = 1, 1, 1 \qquad j = 3$$

$$\therefore y(n) = (A_1 + nA_2 + n^2A_3)(1)^n$$

$$= A_1 + nA_2 + n^2A_3$$

Q:→ Solve the reccurence relation

$$a_{r+4} + 2a_{r+3} + 3a_{r+2} + 2a_{r+1} + a_r = 0$$

sol:→ For char eqn, take $a_r = a^r$

$$a^{r+4} + 2a^{r+3} + 3a^{r+2} + 2a^{r+1} + a^r = 0$$

$$\Rightarrow \quad a^4 + 2a^3 + 3a^2 + 2a + 1 = 0 \qquad \pm 1$$

$$\Rightarrow \quad a^4 + a^3 + a^3 + a^2 + a^2 + a^2 + a + a + 1 = 0$$

$$\Rightarrow \quad \underbrace{a^4 + a^3 + a^2} + \underbrace{a^3 + a^2 + a} + \underbrace{a^2 + a + 1} = 0$$

$$\Rightarrow \quad a^2(a^2 + a + 1) + a(a^2 + a + 1) + (a^2 + a + 1) = 0$$

$$\Rightarrow \quad (a^2 + a + 1)(a^2 + a + 1) = 0$$

$$\Rightarrow \quad (a^2 + a + 1)^2 = 0$$

$$\Rightarrow \quad a^2 + a + 1 = 0 \quad , \quad a^2 + a + 1 = 0$$

$$\Rightarrow \quad a = \frac{-1 \pm i\sqrt{3}}{2} \quad , \quad \frac{-1 \pm i\sqrt{3}}{2}$$

$$a = \frac{-1 + i\sqrt{3}}{2}, \frac{-1 + i\sqrt{3}}{2}, \frac{-1 - i\sqrt{3}}{2}, \frac{-1 - i\sqrt{3}}{2}$$

$$a_r = (A_1 + rA_2)\left(\frac{-1 + i\sqrt{3}}{2}\right)^r + (A_3 + rA_4)\left(\frac{-1 - i\sqrt{3}}{2}\right)^r$$

Q:→ Find the solution of Fibonacci seq

i.e 0, 1, 1, 2, 3, 5, 8, 13, 21, - - . . . .

sol:→ $F_0 = 0$

$F_1 = 1$

$F_2 = 1 = F_0 + F_1$

$F_3 = 2 = F_1 + F_2$

$F_4 = 3 = F_2 + F_3$

$\vdots$

$F_n = F_{n-1} + F_{n-2} \quad , \; n \geq 2$

The recc. relation for the Fibonacci seq is

$$F_n = F_{n-1} + F_{n-2} \quad , \; n \geq 2 \quad \text{—— (1)}$$

with $F_0 = 0$ and $F_1 = 1$

For char eqn take $F_n = a^n$ in (1)

$$a^n = a^{n-1} + a^{n-2}$$

$$a^n - a^{n-1} - a^{n-2} = 0$$

$$\Rightarrow \quad a^2 - a - 1 = 0$$

$$\Rightarrow \quad a = \frac{1 \pm \sqrt{5}}{2}$$

i.e $\quad a = \dfrac{1+\sqrt{5}}{2} \;,\; \dfrac{1-\sqrt{5}}{2}$

$$\therefore \; F_n = A\left(\frac{1+\sqrt{5}}{2}\right)^n + B\left(\frac{1-\sqrt{5}}{2}\right)^n$$

As $F_0 = 0$ and $F_1 = 1$

$A + B = 0$ and $A\left(\dfrac{1+\sqrt{5}}{2}\right) + B\left(\dfrac{1-\sqrt{5}}{2}\right) = 1$

$B = -A$

$$A\left(\frac{1+\sqrt{5}}{2}\right) - A\left(\frac{1-\sqrt{5}}{2}\right) = 1$$

$$\frac{A}{2}\left[(1+\sqrt{5})-(1-\sqrt{5})\right]=1$$

$$\frac{A}{2}(2\sqrt{5})=1$$

$$A=\frac{1}{\sqrt{5}} \Rightarrow B=-\frac{1}{\sqrt{5}}$$

$$\therefore F_n=\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n-\frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

$$=\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n-\left(\frac{1-\sqrt{5}}{2}\right)^n\right] \quad \text{Ans}$$

Ex:→ Solve Recc. Relation

(i) $D(K)-8D(K-1)+12D(K-2)=0$

$D(0)=54$, $D(1)=308$

[Ans $D(K)=4(2)^K+50(6)^K$]

(ii) $S(k)-4S(k-1)-11S(k-2)+30S(k-3)=0$

$S(0)=0$, $S(1)=35$, $S(2)=-85$

[Ans $S(k)=2^k-5^{k+1}+4(-3)^k$]

(iii) $y(k+4)+4y(k+3)+8y(k+2)+8y(k+1)+4y(k)=0$

[Ans

$y(k)=(A_1+kA_2)(-1+i)^k+(A_3+kA_4)(-1-i)^k$